ON ANGLE SUMS AND STEINER POINTS OF POLYHEDRA

BY

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ABSTRACT

In the first part, the Euler-Gram formula for the angle sum of a convex polytope is extended to cellular decompositions of arbitrary polyhedra. The second part contains an attempt to define Steiner points for unions of finitely many convex compact sets, and states some of their properties.

Recently the Euler-Gram formula has been applied successfully to other geomtric questions (see [7]), and it may be natural to look for analogues of this formula in more general contexts. In Proposition 2 below we shall prove a relation between the angle sum of a polyhedron and the Euler characteristic of its interior.

Most of our results about generalized Steiner points are not new. They are contained in the partially unpublished work of G. T. Sallee and G. C. Shephard. Our approach may have the advantage that all calculations are relatively short. They are based on well known properties of the Euler characteristic. I am indebted to the referee for many clarifying comments and hints. For example, the present proofs of Lemma 1 and 3 are entirely his.

Let E^n ($n \geqq 1$) be the n-dimensional Euclidean space. The standard sphere in E^n will be denoted by S^{n-1}, and for each point $p \in E^n$ and each positive $\varepsilon \in \mathbb{R}$, $K_\varepsilon(p)$ denotes the closed ball with center p and radius ε. If X is a set in E^n, we denote by $\underline{X}$ or $\operatorname{int} X$ its interior, by $\bar{X}$ its closure and by $\hat{X}$ its boundary. conv X stands for the convex hull of X. For a convex set K the relative interior relint K is the set of its interior points with respect to the topology of the affine hull aff$K$, and relbd $K = \bar{K}$ — relint K is the relative boundary of K. Let $\mathfrak{K}^n$ be the set of convex compact subsets of E^n and $\mathfrak{Y}^n$ the set of finite unions of elements of $\mathfrak{K}^n$. $\mathfrak{X}^n \subset \mathfrak{Y}^n$ shall contain all sets $X \in \mathfrak{Y}^n$ whose Euler characteristic $\chi(X)$ does not vanish. For a definition of χ, the reader may consult the paper [5] by H. Hadwiger.

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A polyhedron $Q \subset E^n$ is a finite union of convex polytopes, or, equivalently, the union of the cells of a (finite Euclidean) cell complex $\mathfrak{Q}$, $Q = |\mathfrak{Q}| = \bigcup_{X \in \mathfrak{Q}} X$. We eventually say that $\mathfrak{Q}$ is a cellular decomposition of the polyhedron Q. Given a cell complex $\mathfrak{Q}$ we denote its k-skeleton ($k \geqq 0$) by $\mathfrak{Q}^k$, its dimension by $d(\mathfrak{Q})$. and the set of its k-dimensional cells ($k \geqq 0$) by $\Delta^k(\mathfrak{Q})$. For each cell $X \in \mathfrak{Q}$ we have either relint $X \subset \mathfrak{Q} = \text{int}\,|\mathfrak{Q}|$ or $X \subset \hat{Q}$. In the first case we call X an interior cell, in the second case a boundary cell of $\mathfrak{Q}$. The set of boundary cells is a subcomplex $\hat{\mathfrak{Q}}$ of $\mathfrak{Q}$, and we find that $|\hat{\mathfrak{Q}}| = \hat{Q}$.

Now let $\mathfrak{B}$ be any cellular decomposition of the boundary $\hat{Q}$. Given a cell $B \in \mathfrak{B}$ we choose a point $b \in \text{relint}\, B$ and define the interior angle of Q at B with respect to b by

$$\xi(Q,B,b) = \lim_{\varepsilon \downarrow 0} (V(K_\varepsilon(b) \cap Q)/V(K_\varepsilon(b))),$$

where $V(X)$ denotes the volume of X.

LEMMA 1. *$\xi(Q,B,b)$ does not depend on the choice of $b \in$ relint B.*

PROOF. Suppose $a, b \in \text{relint}\, B$. Denote by δ the (positive) distance between the line segment $[a,b]$ and the compact set $\bigcup\{X \in \mathfrak{B} : B \not\subset X\}$ (these two sets are disjoint), and let $W = \bigcup\{\text{aff}\, X : X \in \mathfrak{B} \text{ and } B \subset X\}$. It suffices to show that for each vector $x \in E^n$ of length $< \delta$, $a + x \in Q$, iff $b + x \in Q$. Since W is a set of measure 0, we may restrict our attention to vectors x such that $a + x \notin W$. (Note that if $a + x \notin W$ then $[a + x,\, b + x] \cap W = \emptyset$.) Now suppose $x \in E^n$, $\|x\| < \delta$, and $a + x \notin W$. The segment $[a + x,\, b + x]$ intersects no cell X of $\mathfrak{B}$, hence $[a + x,\, b + x] \cap \hat{Q} = \emptyset$. It follows that either $[a + x,\, b + x] \subset Q$, or $[a + x,\, b + x] \cap Q = \emptyset$. Q.E.D.

In view of Lemma 1 we can write $\xi(Q,B)$ instead of $\xi(Q,B,b)$. Now let $\mathfrak{X}$ be a subcomplex of a cellular decomposition $\mathfrak{B}$ of $\hat{Q}$. We set $\alpha_k(Q,\mathfrak{X}) = \Sigma_{B \in \Delta^k(\mathfrak{X})}\xi(Q,B)$ and define the (interior) angle sum of Q with respect to X by

$$\alpha(Q,\mathfrak{X}) = \sum_{k=0}^{d(\mathfrak{X})} (-1)^k \alpha_k(Q,\mathfrak{X}).$$

The classical Euler-Gram relation can be stated as follows.

PROPOSITION 1. (*Euler-Gram*). *If $Q \subset E^n$ is a convex polytope with non-empty interior and $\mathfrak{B}$ is its natural boundary complex then $\alpha(Q,\mathfrak{B}) = (-1)^{n-1}$.*

Several independent proofs of this equality have appeared; see, for example, [6], [4], ch. 14, or [1], where the main result is a Gauss-Bonnet formula for curvilinear polytopes. Proposition 2 below shows a certain analogy to this formula, since in both the Euler characteristic of the interior of the considered set appears, but we do not know exactly how far this analogy reaches. In the case of convex polytopes in E^3, known already to R. Descartes [2], the two statements can easily be shown to be equivalent.

PROPOSITION 2. *For each polyhedron* $Q \subset E^n$ *and each cellular decomposition* $\mathfrak{B}$ *of its boundary,* $\alpha(Q,\mathfrak{B}) = (-1)^{n-1}\chi(\operatorname{int} Q)$, *where* χ *denotes the Euler characteristic.*

The proof depends on two lemmas. First let us show that α is invariant under subdivisions.

LEMMA 2. *For each subdivision* $\mathfrak{C}$ *of* $\mathfrak{B}$, $\alpha(Q,\mathfrak{C}) = \alpha(Q,\mathfrak{B})$.

PROOF. If $\mathfrak{X}$ is a subcomplex of $\mathfrak{B}$ we denote by $\mathfrak{Y}$ the subcomplex of $\mathfrak{C}$ which subdivides $\mathfrak{X}$. We want to prove, for each $\mathfrak{X} \subset \mathfrak{B}$, that $\alpha(Q,\mathfrak{X}) = \alpha(Q, \mathfrak{Y})$. This is obviously true if the dimension $d(\mathfrak{X})$ is zero. Given a complex $\mathfrak{X}$ with $d(\mathfrak{X}) = k$ let $h(\mathfrak{X})$ be the number of k-cells in $\mathfrak{X}$. We assume our assertion to be true for all complexes $\mathfrak{X}' \subset \mathfrak{B}$ with $d(\mathfrak{X}') < d(\mathfrak{X}) = k$ or $d(\mathfrak{X}') = d(\mathfrak{X}) = k$ and $h(\mathfrak{X}') < h(\mathfrak{X})$, and want to prove it for $\mathfrak{X}$.

First case, $h(\mathfrak{X}) = 1$. $\mathfrak{X}$ contains a single k-cell X_0. Set $\mathfrak{X}_1 = \mathfrak{X} - \{X_0\}$ and let $\mathfrak{Y}_1$ be the corresponding subcomplex of $\mathfrak{Y}$. We have $\alpha_i(Q,\mathfrak{X}) = \alpha_i(Q,\mathfrak{X}_1)$ $(0 \leqq i \leqq k-1)$ and $\alpha_k(Q,\mathfrak{X}) = \xi(Q,X_0)$, $\alpha_k(Q,\mathfrak{X}_1) = 0$. The equations for $\mathfrak{Y}$ and $0 \leqq i \leqq k-1$ are $\alpha_i(Q,\mathfrak{Y}) = \alpha_i(Q,\mathfrak{Y}_1) + \xi(Q,X_0)(f^i(\mathfrak{Y}) - f^i(\mathfrak{Y}_1))$, whereas $\alpha_k(Q,\mathfrak{Y}) = \xi(Q,X_0)f^k(\mathfrak{Y}), \alpha_k(Q,\mathfrak{Y}_1) = f^k(\mathfrak{Y}_1) = 0$. Here $f^i(\mathfrak{A})$ stands for the number $\operatorname{card}\Delta^i(\mathfrak{A})$ of i-cells in the complex $\mathfrak{A}$. Consequently we have $\alpha(Q,\mathfrak{Y}) = \alpha(Q,\mathfrak{Y}_1) + \xi(Q,X_0)(\chi(|\mathfrak{Y}|) - \chi(|\mathfrak{Y}_1|))$. Since $|\mathfrak{Y}_1| \cup X_0 = |\mathfrak{Y}|$ and $|\mathfrak{Y}_1| \cap X_0 = \operatorname{relbd} X_0$ we find, by the valuation property of χ, $\chi(|\mathfrak{Y}|) - \chi(|\mathfrak{Y}_1|) = \chi(X_0) - \chi(\operatorname{relbd} X_0)$, which gives us $\alpha(Q,\mathfrak{Y}) = \alpha(Q,\mathfrak{Y}_1^k) + (-1)^k\xi(Q,X_0)$. On the other hand we have $\alpha(Q,\mathfrak{X}) = \alpha(Q,\mathfrak{X}_1) + (-1)^k\xi(Q,X_0)$. Applying the inductive hypothesis to the complexes $\mathfrak{X}_1$ and $\mathfrak{Y}_1$ we find indeed that $\alpha(Q,\mathfrak{X}) = \alpha(Q,\mathfrak{Y})$.

Second case, $h(\mathfrak{X}) > 1$. Choose a k-cell $X_0 \in \mathfrak{X}$ and let $\mathfrak{X}_1, \mathfrak{Y}_1$ be defined as in the first case. Denote by $\mathfrak{X}_0$ the complex consisting of X_0 and all its faces and by $\mathfrak{Y}_0$ the corresponding subdivision. We have

$$\alpha_i(Q,\mathfrak{X}_0) + \alpha_i(Q,\mathfrak{X}_1) = \alpha_i(Q,\mathfrak{X}) + \alpha_i(Q,\mathfrak{X}_0 \cap \mathfrak{X}_1)$$

$$\alpha_i(Q,\mathfrak{Y}_0) + \alpha_i(Q,\mathfrak{Y}_1) = \alpha_i(Q,\mathfrak{Y}) + \alpha_i(Q,\mathfrak{Y}_0 \cap \mathfrak{Y}_1),$$

and the corresponding equations for α. By the inductive assumption we find $\alpha(Q,\mathfrak{X}_0) = \alpha(Q,\mathfrak{Y}_0)$, $\alpha(Q,\mathfrak{X}_1) = \alpha(Q,\mathfrak{Y}_1)$ and $\alpha(Q,\mathfrak{X}_0 \cap \mathfrak{X}_1) = \alpha(Q,\mathfrak{Y}_0 \cap \mathfrak{Y}_1)$, from which the desired assertion follows.

The next lemma allows us to "blow up towards the interior" the highest dimensional cells of a complex.

LEMMA 3. *Let $\mathfrak{Q}$ be a cell complex, $Q = |\mathfrak{Q}| \subset E^n$, and choose an n-dimensional cell $Q_0 \in \mathfrak{Q}$. Then there exists a convex polytope R such that*

1) $Q_0 \subset R \subset Q$;
2) *each point of Q_0 which is an interior point of Q is an interior point of R.*

PROOF. Each cell X of $\hat{\mathfrak{Q}}$ intersects Q_0 in a (possibly empty) proper face of Q_0. For each $X \in \hat{\mathfrak{Q}}$ there exists a hyperplane H_X such that $Q_0 \subset H_X^+$, $X \subset H_X^-$, $Q_0 \cap H_X = X \cap H_X = Q_0 \cap X$, where H_X^+ and H_X^- are the two closed half spaces determined by H_X. Choose such a hyperplane H_X for each $X \in \hat{\mathfrak{Q}}$, and let $R = \bigcap \{H_X^+ : X \in \hat{\mathfrak{Q}}\}$. R is a polyhedral convex set, and clearly $Q_0 \subset R$. If $q \in Q_0$ is an interior point of Q, then $q \notin H_X$ for all $X \in \hat{\mathfrak{Q}}$, hence $q \in R$. We still have to show that $R \subset Q$. ($R \subset Q$ implies that R is bounded, hence a convex polytope.) Suppose $R \not\subset Q$, and choose a point $x \in R - Q$. If $y \in \underline{Q}_0 \subset Q$ then $y \in \underline{R}$, and there exists a point $z \in [x, y] \cap \hat{Q}$, $z \neq x$. It follows that $z \in \underline{R} \cap \hat{Q}$. But $\underline{R} \cap \hat{Q} = \emptyset$, by the construction of R, a contradiction. Q.E.D.

Now we are ready for the

PROOF OF PROPOSITION 2. Let $Q = |\mathfrak{Q}|$ and $\mathfrak{B}$ be given. We want to establish Proposition 2 by induction on the numbetr $h(\mathfrak{Q})$ of n-dimensional cells in $\mathfrak{Q}$.

The case $h(\mathfrak{Q}) = 0$ is trivial if we notice that $Q = \emptyset$ implies that $\alpha(Q,\mathfrak{B}) = 0$. In the case $h(\mathfrak{Q}) = 1$ let Z be the n-cell contained in $\mathfrak{Q}$ and denote by $\mathfrak{Z}$ the set of all cells $X \subset \mathfrak{B}$ which are subsets of $\hat{Z}$. $\mathfrak{Z}$ is a cell complex, and $|\mathfrak{Z}| \subset \hat{Z}$. On the other hand, let p be a point of $\hat{Z}$ and $X_p \in \mathfrak{B}$ the cell to whose relative interior p belongs. We have $\xi(Q, X_p, p) \neq 0$, and by Lemma 1 the angle $\xi(Q, X_p, q)$ is different from zero for each point $q \in \operatorname{relint} X_p$. This implies $q \in \hat{Z}$ and $X_p \subset \hat{Z}$. Since p was chosen arbitrarily we find $\hat{Z} \subset |\mathfrak{Z}|$. So $\mathfrak{Z}$ is a cellular decomposition of $\hat{Z}$. For each $X \in \mathfrak{B} - \mathfrak{Z}$ we find, again by Lemma 1, $\xi(Q, X) = 0$, and consequently $\alpha(Q,\mathfrak{B}) = \alpha(Q,\mathfrak{Z}) = \alpha(Z,\mathfrak{Z})$. $\mathfrak{Z}$ is a subdivision of the natural boundary complex of the convex polytope Z, therefore, by Proposition 1 and Lemma 2, we find $\alpha(Z,\mathfrak{Z}) = (-1)^{n-1}$, which establishes the case $h(\mathfrak{Q}) = 1$. Assume then $h(\mathfrak{Q}) > 1$ and suppose that Proposition 2 has been proved for all polyhedra

$Q' = |\mathfrak{Q}'|$ with $h(\mathfrak{Q}') < h(\mathfrak{Q})$. We choose an n-cell $Q_0 \in \mathfrak{Q}$. By Lemma 3 there is a convex polytope R such that $Q_0 \subset R \subset Q$, and each point $p \in Q_0 \cap \underline{Q}$ belongs to $\underline{R}$. Let $\mathfrak{R}$ be the complex consisting of R and all its faces and set $\mathfrak{S} = \mathfrak{Q} - \{Q_0\}$. The set $\mathfrak{D}$ of all intersections $X \cap Y$ $(X \in \mathfrak{R},\ Y \in \mathfrak{S})$ is a cell complex. Denoting by $S = |\mathfrak{S}|$ and $D = |\mathfrak{D}|$ the corresponding polyhedra we find $\underline{R} \cup \underline{S} = \underline{Q}$, $\underline{R} \cap \underline{S} = \underline{D}$. From this we deduce

$$\hat{R} \cup \hat{S} = \hat{Q} \cup \hat{D},\ \hat{R} \cap \hat{S} = \hat{Q} \cap \hat{D}. \tag{3}$$

We choose a cellular decomposition $\mathfrak{C}$ of $\hat{R} \cup \hat{S}$ such that there are subcomplexes $\mathfrak{C}_Q$, $\mathfrak{C}_D$, $\mathfrak{C}_R$, $\mathfrak{C}_S$ which decompose $\hat{Q}$, $\hat{D}$, $\hat{R}$ and $\hat{S}$ respectively. (3) implies

$$\mathfrak{C}_R \cup \mathfrak{C}_S = \mathfrak{C}_Q \cup \mathfrak{C}_D = \mathfrak{C},\ \mathfrak{C}_R \cap \mathfrak{C}_S = \mathfrak{C}_Q \cap \mathfrak{C}_D. \tag{4}$$

Let us show that for each k, $0 \leqq k \leqq n$,

$$\alpha_k(R, \mathfrak{C}_R) + \alpha_k(S, \mathfrak{C}_S) = \alpha_k(Q, \mathfrak{C}_Q) + \alpha_k(D, \mathfrak{C}_D). \tag{5}$$

Consider a cell X of dimension k in $\mathfrak{C}$ an a point $x \in \operatorname{relint} X$. First case, $X \in \mathfrak{C}_R$, $X \notin \mathfrak{C}_S$. This means that $x \in \underline{S}$, since $\hat{R} \subset S$. There exists an $\varepsilon > 0$ such that $K_\varepsilon(x) \subset S$. Since $\underline{S} \subset \underline{Q}$ we find that $x \notin \hat{Q}$ and $X \notin \mathfrak{C}_Q$, but on the other hand, considering the first equation in (4), we have $X \in \mathfrak{C}_D$. For every $\mu \leqq \varepsilon$ we obtain $K_\mu(x) \cap D = K_\mu(x) \cap S \cap R = K_\mu(x) \cap R$, and therefore $\xi(R, X) = \xi(D, X)$. So the contributions of X to $\alpha_k(R, \mathfrak{C}_R)$ and to $\alpha_k(D, \mathfrak{C}_D)$ are equal, whereas X contributes neither to $\alpha_k(S, \mathfrak{C}_S)$ nor to $\alpha_k(Q, \mathfrak{C}_Q)$.

The case $X \notin \mathfrak{C}_R$, $X \in \mathfrak{C}_S$ and $X \in \mathfrak{C}_R$, $X \in \mathfrak{C}_S$ are equally easy to deal with. (5) follows by summation over all k-cells in $\mathfrak{C}$, and we further deduce

$$\alpha(R, \mathfrak{C}_R) + \alpha(S, \mathfrak{C}_S) = \alpha(Q, \mathfrak{C}_Q) + \alpha(D, \mathfrak{C}_D). \tag{6}$$

Since $h(\mathfrak{R}) = 1 < h(\mathfrak{Q})$, the inductive assumption gives us

$$\alpha(R, \mathfrak{C}_R) = (-1)^{n-1}\chi(\underline{R}),$$

and similar expressions for S and D. By the two equations preceding (3) and by the valuation property of χ we have $\chi(\underline{R}) + \chi(\underline{S}) = \chi(\underline{Q}) + \chi(\underline{D})$, which, together with (6), implies $\alpha(Q, \mathfrak{C}_Q) = (-1)^{n-1}\chi(\underline{Q})$. Given any cellular decomposition $\mathfrak{B}$ of $\hat{Q}$, there exists a common subdivision $\mathfrak{U}$ of $\mathfrak{B}$ and $\mathfrak{C}_Q$, and Lemma 2 guarantees the equality $\alpha(Q, \mathfrak{C}_Q) = \alpha(Q, \mathfrak{U}) = \alpha(Q, \mathfrak{B})$, from which Proposition 2 follows. In the case $\underline{Q} = \varnothing$ our formula is without interest, and we have not been able, so far, to define interior angles for a polyhedron $Q \subset E^n$ of lower dimension.

We only remark that there are polyhedra Q' such that Q is a deformation retract of Q', and for each such Q' we have $\alpha(Q') = (-1)^{n-1}\chi(Q)$.

Let $K \subset E^n$ be a compact convex set. In [9] its Steiner point has been defined as $s(K) = (1/\omega_n) \int_{S^{n-1}} \pi(K,u)u\,d\omega$, where $\pi(K,u)$ is the value of the support function of K in the direction $u \in S^{n-1}$, $d\omega$ denotes the volume element of S^{n-1} and ω_n stands for the integral $\int_{S^{n-1}}(a,u)^2 d\omega$, a being any constant vector on S^{n-1}. G. T. Sallee [8] defines a "Steiner point" $\hat{s}(Y)$ for arbitrary sets $Y \in \mathfrak{Y}^n$. When restricted to the class of those sets $Y \in \mathfrak{Y}^n$ whose Euler characteristic $\chi(Y)$ is one, $\hat{s}$ behaves covariantly under similarity transformations, and has the valuation property. Sallee does not only prove this, but also remarks that $\hat{s}$ has the valuation property for arbitrary members of $\mathfrak{Y}^n$. Using H. Hadwiger's recursive formula ([5] formula (11)) for the Euler characteristic χ, we give a different definition of $\hat{s}$, which immediately allows us to derive Sallee's remark from the corresponding property of χ. We describe the influence of similarity transformations on $\hat{s}(Y)$ ($Y \in \mathfrak{Y}^n$), and define, for all $X \in \mathfrak{X}^n$, a point $t(X)$, which is closely related to $\hat{s}(X)$ and behaves covariantly under similarities. Both $\hat{s}$ and t reflect some properties of the classical Steiner point, but neither of them fulfills all our wishes in this direction. As for a seemingly quite different generalization, where only the difficulties are somehow similar to the ones we met, see H. Flanders [3].

Given a direction $u \in S^{n-1}$ and a number $\lambda \in \mathbb{R}$, denote by $H(\lambda, u)$ the hyperplane which contains λu and is perpendicular to the line through the origin containing u. For each

$$Y \in \mathfrak{Y}^n \text{ set } \delta(Y,u,\lambda) = \chi(Y \cap H(\lambda,u)) - \lim_{\mu \downarrow \lambda} \chi(Y \cap H(\mu,u)),$$

where $\lim_{\mu \downarrow \lambda}$ indicates that we only consider the numbers $\mu > \lambda$. If Y is convex and $\pi(Y,u)$ denotes the value of the support function of Y in the direction u, we have

$$\delta(Y,u,\lambda) = 0 \text{ for } \lambda \neq \pi(Y,u) \text{ and } \delta(Y,u,\pi(Y,u)) = 1. \tag{7}$$

So, for arbitrary $Y \in \mathfrak{Y}^n$, there are only finitely many values of λ for which $\delta(Y,u,\lambda) \neq 0$, and we can define the vector sum $\sigma(Y,u) = \sum_{\lambda \in \mathbb{R}} \delta(Y,u,\lambda)\lambda u$.

Lemma 4. *For each $Y \in \mathfrak{Y}^n$ the integral $\hat{s}(Y) = (1/\omega_n) \int_{S^{n-1}} \sigma(Y,u)d\omega$ exists, and if Y is convex then $\hat{s}(Y) = s(Y)$.*

Proof. By (7) we find, for a convex set $Y \in \mathfrak{Y}^n$, $\sigma(Y,u) = \pi(Y,u)u$ and therefore $\hat{s}(Y) = s(Y)$. For arbitrary $Y \in \mathfrak{Y}^n$ there are convex sets $K_1, \cdots, K_r \in \mathfrak{Y}^n$ such that $Y = K_1 \cup \cdots \cup K_r$, and by the valuation property of χ the equation

$$(8)\ \sigma(Y,u) = \sum_{i=1}^{r} \sigma(K_i,u) - \sum_{i<j} \sigma(K_i \cap K_j, u) + \cdots + (-1)^{r-1}\sigma(K_1 \cap \cdots \cap K_r, u)$$

follows. $\sigma(Y,u)$, being a finite sum of integrable functions, is itself integrable.

For $X \in \mathfrak{X}^n \subset \mathfrak{Y}^n$ we set $t(X) = \hat{s}(X)/\chi(X)$, which again coincides with the Steiner point $s(X)$ if X is convex.

PROPOSITION 3. *t behaves covariantly under similarity transformations.*

PROOF. If $\alpha: E^n \to E^n$ is an orthogonal transformation we easily verify that $\sigma(\alpha X, \alpha u) = \alpha\sigma(X,u)$, and consequently $t(\alpha X) = \alpha t(X)$. If α: $E^n \to E^n$ is a dilatation with a positive factor we find that $\sigma(\alpha X, u) = \alpha\sigma(X,u)$ and therefore $t(\alpha X) = \alpha t(X)$. The same still holds if the factor of α is zero, since obviously $t(\{0\}) = 0$. Now let α be a translation, $\alpha(X) = X + b$. We find

$$\delta(X + b, u, \lambda + (b,u)) = \delta(X, u, \lambda)$$

and therefore $\sigma(X + b, u) = \sigma(X,u) + (b,u)(\sum_{\lambda \in \mathbb{R}} \delta(X,u,\lambda))u$. By H. Hadwiger's recursion formula ([5] formula (11)) the sum in brackets is the Euler characteristic $\chi(X)$. So we have $\hat{s}(X + b) = \hat{s}(X) + \chi(X)(1/\omega_n) \int_{S^{n-1}}(b,u)u\,d\omega$, or, since the last integral gives us the Steiner point $s(\{b\}) = b$, $\hat{s}(X + b) = \hat{s}(X) + \chi(X)b$, which proves Proposition 3.

The valuation property of Steiner points is a corollary of the next statement.

PROPOSITION 4. *Let Y_i $(1 \leqq i \leqq r)$ be sets in $\mathfrak{Y}^n$ and set $Y = Y_1 \cup \cdots \cup Y_r$. Then*

$$\hat{s}(Y) = \sum_{i=1}^{r} \hat{s}(Y_i) - \sum_{i<j} \hat{s}(Y_i \cap Y_j) + \cdots + (-1)^{r-1}\hat{s}(Y_1 \cap \cdots \cap Y_r).$$

PROOF. This follows immediately from the corresponding formula (8) for $\sigma(Y,u)$. We only have to notice that (8) still holds if we replace the convex sets K_i by arbitrary elements Y_i of $\mathfrak{Y}^n$.

Sometimes the points $t(X)$ and $t(\hat{X})$ coincide.

LEMMA 5. *For each convex polytope $K \in \mathfrak{K}^n$,*

$$\hat{s}(\operatorname{relbd} K) = \chi(\operatorname{relbd} K)\hat{s}(K).$$

PROOF. Denote by E^d the affine hull of K. Lemma 5 is true in the case $d = 0$, since $\hat{s}(\emptyset) = 0$. So let us assume $d > 0$. The set T of directions u for which $H(1,u)$ contains a flat parallel to E^d is a proper subsphere S^{n-d-1} of S^{n-1} and can be neglected in the integration. For each direction $u \in S^{n-1} - T$ we find

$\delta(\text{relbd}\, K, u, \lambda) = 0$ if $\lambda \notin \{\pi(K,u), -\pi(K,-u)\}$ and $\delta(\text{relbd}\, K, u, \pi(K,u)) = 1$, $\delta(\text{relbd}\, K, u, -\pi(K,-u) = 1 - \chi(S^{d-2})$.

Summation over λ gives

$$\sigma(\text{relbd}\, K, u) = \pi(K,u)u + (-\pi(K,-u))(1 - \chi(S^{d-2}))u.$$

So we have

$$2\omega_h \hat{s}(\text{relbd}\, K) = \int_{S^{n-1}} (\sigma(\text{relbd}\, K, u) + \sigma(\text{relbd}\, K, -u))d\omega$$

$$= (2 - \chi(S^{d-2}))\left(\int_{S^{n-1}} \pi(K,u)ud\omega + \int_{S^{n-1}} \pi(K,-u)d\omega\right),$$

and finally $\hat{s}(\text{relbd}\, K) = \chi(S^{d-1})\hat{s}(K)$.

We conclude with a formula, which relates the Steiner points of the different cells of a cell complex to each other. Let $\mathfrak{P}$ be a cell complex and $P = |\mathfrak{P}|$. Denote by $\hat{s}_i(\mathfrak{P})$ the sum $\Sigma_{Z \in \Delta^i(\mathfrak{P})} \hat{s}(Z)$.

Proposition 5. *For each cell complex* $\mathfrak{P}$, *with* $P = |\mathfrak{P}|$, *the relation* $\hat{s}(P) = \Sigma_{i=0}^{d(\mathfrak{P})}(-1)^i \hat{s}_i(\mathfrak{P})$ *holds.*

Proof. The case $d(\mathfrak{P}) = 0$ is an immediate corollary of Proposition 4. Let us then assume, for a given $\mathfrak{P}$, that our statement is true for all complexes $\mathfrak{P}'$ with $d(\mathfrak{P}') < d(\mathfrak{P}) = k$ or $d(\mathfrak{P}') = d(\mathfrak{P}) = k$ and $h(\mathfrak{P}') < h(\mathfrak{P})$, where $h(\mathfrak{P})$ is the number of k-dimensional cells of $\mathfrak{P}$. Choose a k-cell $Z \in \mathfrak{P}$, denote by $\mathfrak{Z}$ the complex consisting of Z and all its faces, and by $\hat{\mathfrak{Z}} = \mathfrak{Z} - \{Z\}$ its natural boundary complex. Set $\mathfrak{Q} = \mathfrak{P} - \{Z\}$ and $Q = |\mathfrak{Q}|$. We have $\mathfrak{Q} \cap \mathfrak{Z} = \hat{\mathfrak{Z}}$ and by Proposition 4 $\hat{s}(Q) + \hat{s}(Z) = \hat{s}(P) + \hat{s}(\text{relbd}\, Z)$. The inductive assumption gives us $\hat{s}(Q) = \Sigma_{i=0}^{d(\mathfrak{P})}(-1)^i \hat{s}_i(\mathfrak{Q})$, and by Lemma 5 we find $\hat{s}(\text{relbd}\, Z) = (1 + (-1)^{k-1})\hat{s}(Z)$. From this we derive the relation

$$\hat{s}(P) = \sum_{i=0}^{d(\mathfrak{P})} (-1)^i \hat{s}_i(\mathfrak{Q}) + (-1)^k \hat{s}(Z),$$

and our Proposition follows, if we look at the definition of $\mathfrak{Q}$.

Taking for $\mathfrak{P}$ the natural boundary complex of a d-dimensional polytope $P \subset E^n$ we have, by Lemma 5, Proposition 5 and the fact that $\hat{s}(P) = s(P)$ for $P \in \mathfrak{K}^n$, the equation $(1 + (-1)^{d-1})s(P) = \Sigma_{i=0}^{d-1}(-1)^i(\Sigma_{F \in \Delta_i(P)} s(F))$, which has been discovered by G. C. Shephard in (9).

Although I have not found Proposition 5 in the literature, I would like to add here that the priority for it, as well as for the formula (8), belongs to Shephard, who, as I learned from the referee, has known them for several years.

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